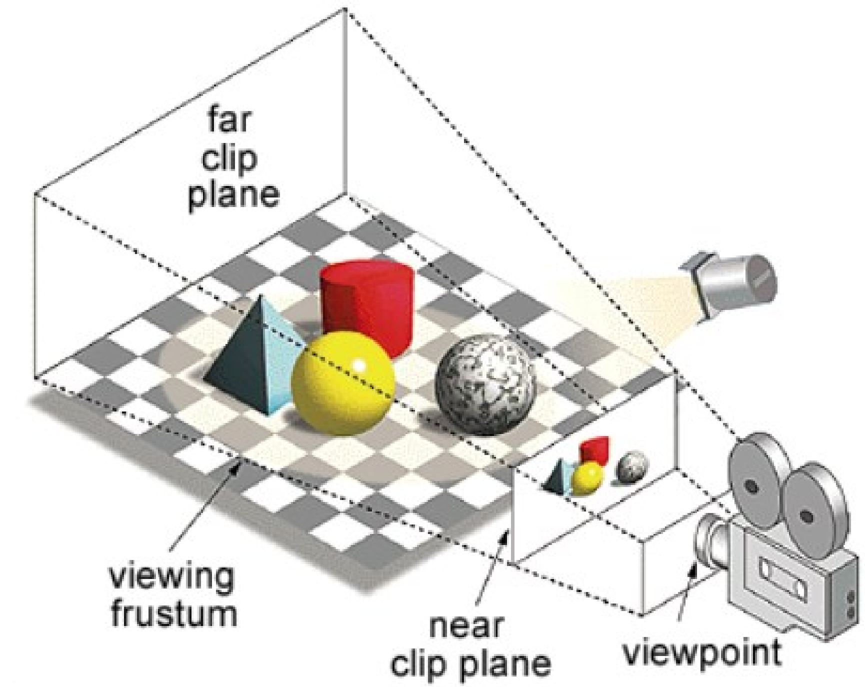
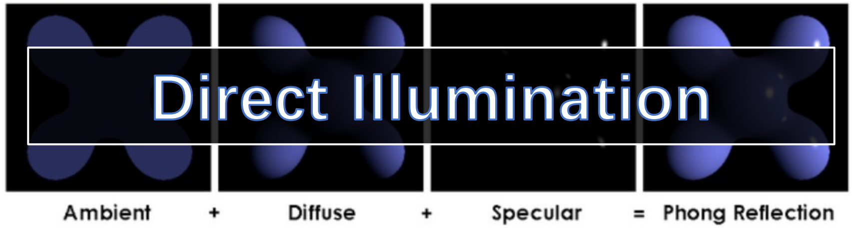
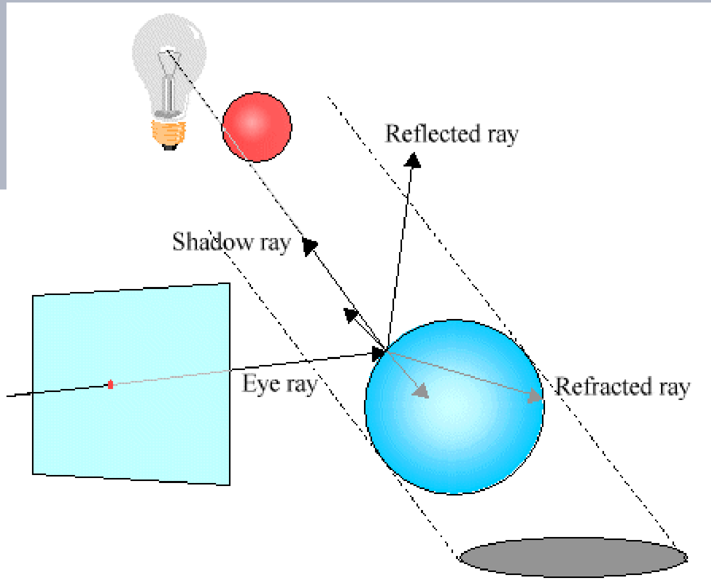
Global Illumination II
Path Tracing

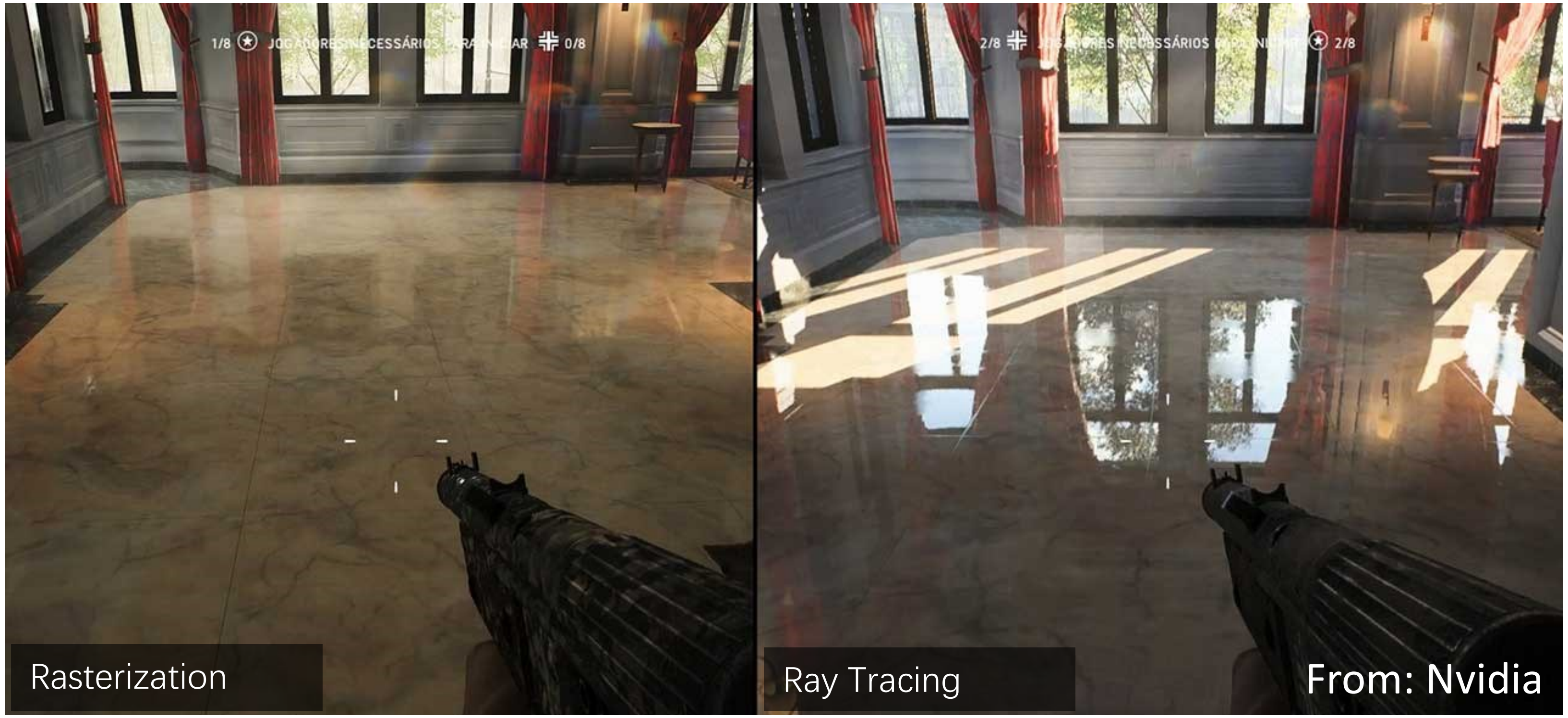
First Question: Why?



$$L = L_a + L_d + L_s$$
$$= k_a I_a + k_d \left(I/r^2\right) \max(0, \mathbf{n}\cdot\mathbf{l}) + k_s \left(I/r^2\right) \max(0, \mathbf{n}\cdot\mathbf{h})^p$$







First Question: Why?

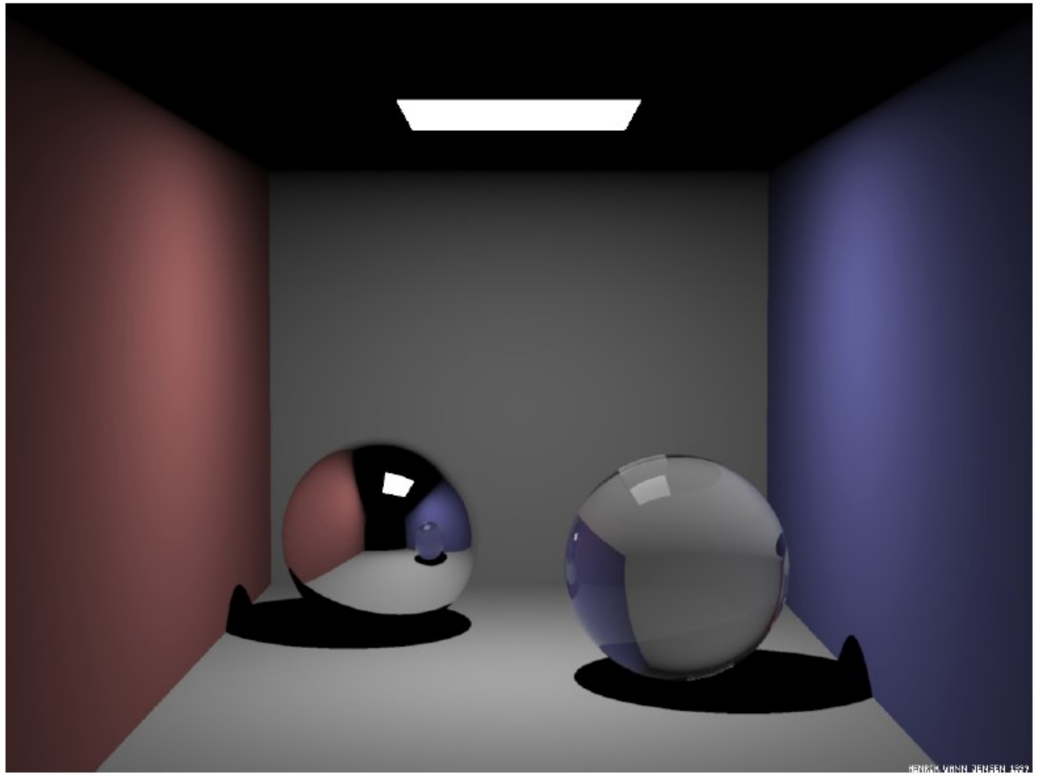
- Can Whitted-Style Ray Tracing Produce the Glossy Effect?

Mirror reflection

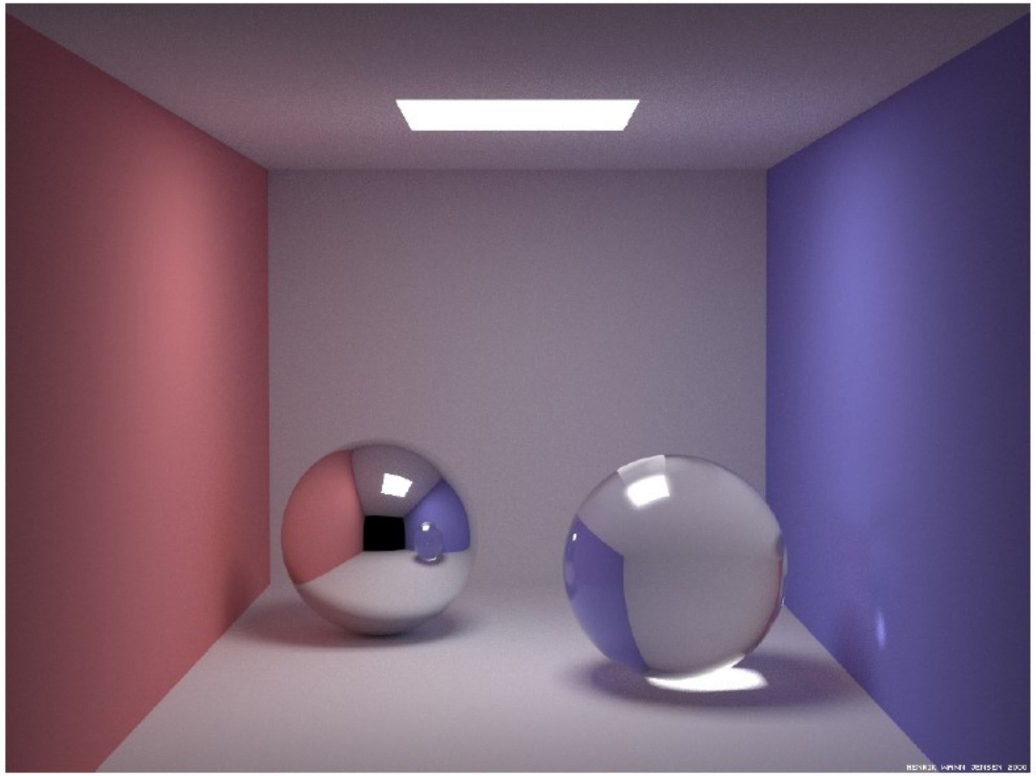
Glossy reflection

First Question: Why?

- How about diffusion reflection?

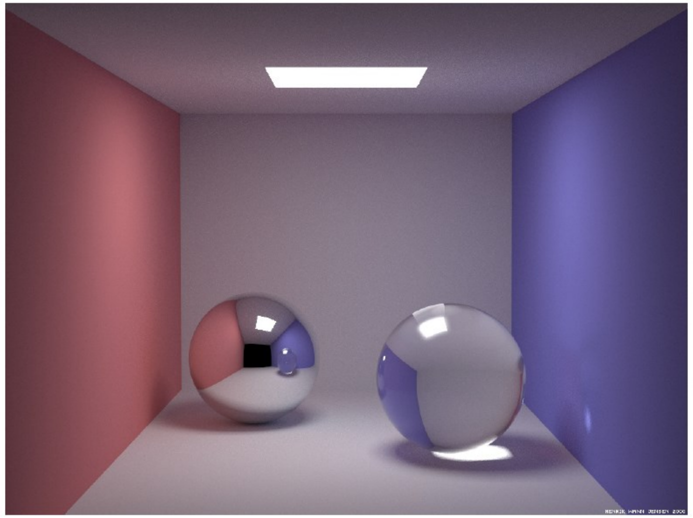


(a) Whitted-Style Ray Tracing



(b) Global Illumination using Photon Maps

First Question: Why?



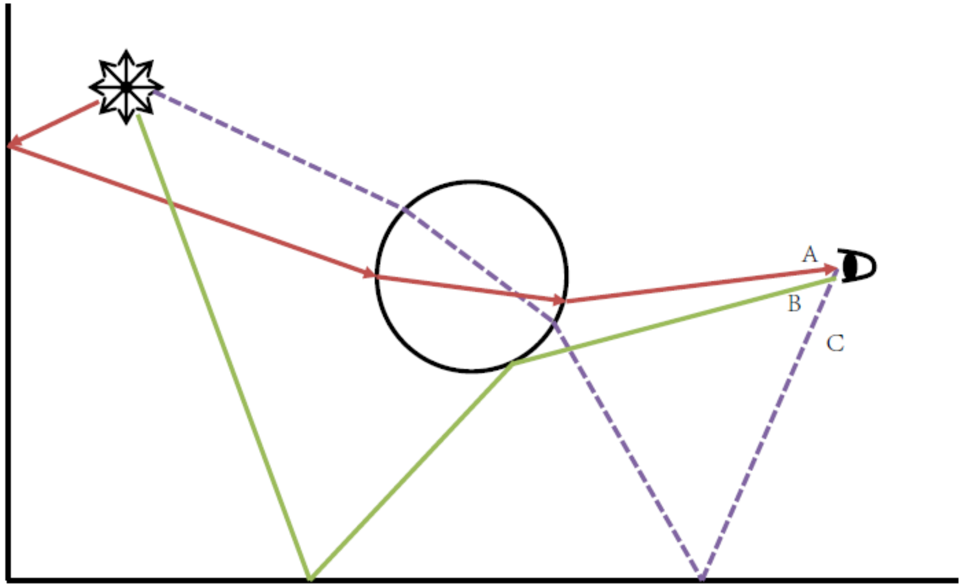
光路传输记法:

L (Light)：光路从光源发出

D (Diffuse)：在物体表面发生漫反射

S (Specular)：在物体表面发生镜面反射或者折射

V (Volumetric Scattering)：进入介质发生体积散射



a,b,c 三条光路，对应记号分别为 LDSSE,LDSE,LSSDE

LDE，LSE	**直接光照**，表面亮度可用传统的直接光照模型来计算，如Phong[11]，Blinn[12] 等。
L(S|D)S+E	一次或多次**镜面反射/折射的间接光照**，如看到的镜子中的物体。经典的光线跟踪算法可以准确计算
LS+ DE	焦散效果(Caustics)，例如玻璃杯投射在桌上的亮斑，是典型的**间接光照**效果之一。
L(S|D)*DDE	漫反射材质间的颜色扩散效果(Color Bleeding)，典型的间接光照效果之一。
L(S|D)+DS+E	被一次或多次镜面反射的焦散或者颜色扩散效果。
L(S|D)+D(S|D)*E	其他的光路传输。由于经过了多次漫反射，其光照强度的贡献往往较低，不易被注意到。

Global Illumination = ALL!

Second Question: What?

Ray Tracing:

- **Whitted** Ray Tracing
- **Distribution** ray tracing:
  - **Path Tracing,** a ray tracing method based **on integral transport equation** and **rendering equation**
    - Path Tracing
    - Bidirectional Path Tracing
    - Metropolis Light Transport
    - Energy Redistribution Path Tracing
    - …

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THE RENDERING EQUATION

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Last Question: How?

- Path Tracing:
  - Radiometry (辐射度量学), a measurement system for illumination
  - **Integral** Light Transport **Equations**
    - The reflection equation
    - The rendering equation
  - **Probability** and Monte Carlo Integration
  - Algorithm

Radiometry

- Radiant Energy Q, Radiant Flux Φ, Radiant Intensity $I(\omega) \equiv \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}$

LED灯: 90 lm/W
白炽灯： 12 lm/W

SI radiometry units

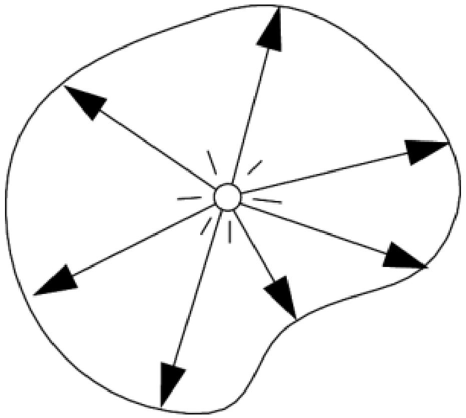
V • T • E

Quantity		Unit		Dimension	Notes
Name	Symbol[nb 1]	Name	Symbol	Symbol	
Radiant energy	Q_e[nb 2]	joule	J	$\mathsf{M}{\cdot}\mathsf{L}^2{\cdot}\mathsf{T}^{-2}$	Energy of electromagnetic radiation.
Radiant flux	Φ_e[nb 2]	watt	W = J/s	$\mathsf{M}{\cdot}\mathsf{L}^2{\cdot}\mathsf{T}^{-3}$	Radiant energy emitted, reflected, transmitted or received, per unit time. This is sometimes also called "radiant power", and called luminosity in Astronomy.
Radiant intensity	$I_{e,\Omega}$[nb 5]	watt per steradian	W/sr	$\mathsf{M}{\cdot}\mathsf{L}^2{\cdot}\mathsf{T}^{-3}$	Radiant flux emitted, reflected, transmitted or received, per unit solid angle. This is a *directional* quantity.

https://en.wikipedia.org/wiki/Radiant_intensity

Radiometry

- Radiant Energy Q, Radiant Flux Φ, Radiant Intensity $I(\omega) \equiv \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}$



Radiant Flux = 815 lumens

Radiant Intensity = 815 lumens / (4π)

SI radiometry units

V • T • E

Quantity		Unit		Dimension	Notes
Name	**Symbol**[nb 1]	**Name**	**Symbol**	**Symbol**	
Radiant energy	Q_e[nb 2]	joule	J	**M**⋅**L**^{2}⋅**T**$^{-2}$	Energy of electromagnetic radiation.
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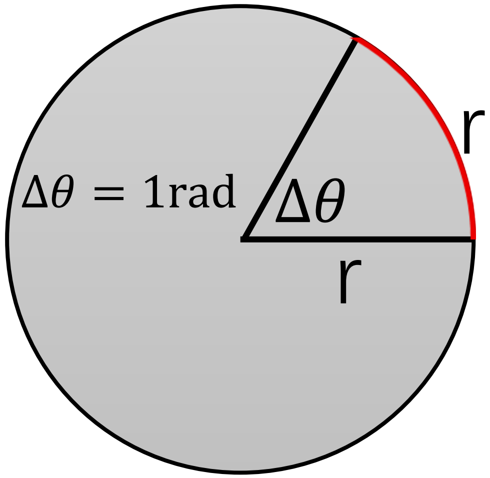
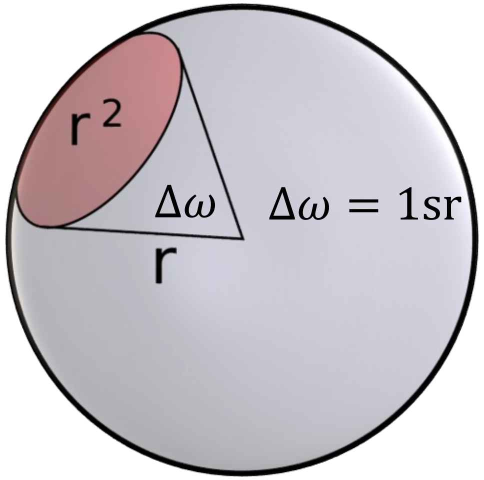
https://en.wikipedia.org/wiki/Radiant_intensity

Radiant Intensity

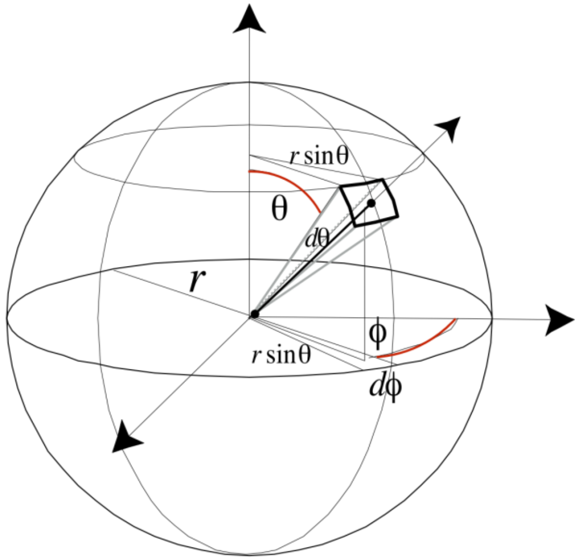
- $I(\omega) \equiv \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}$
- unit W/sr, or cd=candela=lm/sr
- Power **per unit solid angle**
- Solid Angle: $\Delta\omega = \frac{A}{r^2}$ sr
- Sphere has 4π steradians (球面度)





- Along the direction of (θ, ϕ), calculate the $\mathrm{d}\omega$ using $\mathrm{d}\theta, \mathrm{d}\phi$

  Hint: $\mathrm{d}\omega = \frac{\mathrm{d}A}{r^2}$



$$\mathrm{d}\omega = \frac{\mathrm{d}A}{r^2}$$

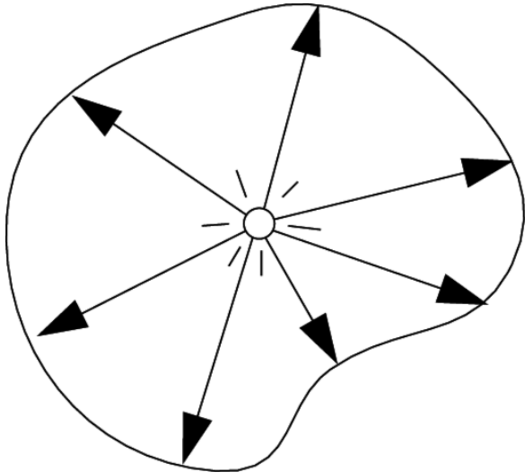
$$\mathrm{d}A = (r\,\mathrm{d}\theta)(r\sin\theta\,\mathrm{d}\phi) = r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$$

$$\mathrm{d}\omega = \sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$$

We can represent $\Delta\omega$ as $(\Delta\theta, \Delta\phi)$, i.e., we can represent ω as (θ, ϕ)

Radiometry

- Important Light Measurements of Interest



Light Emitted From A Source

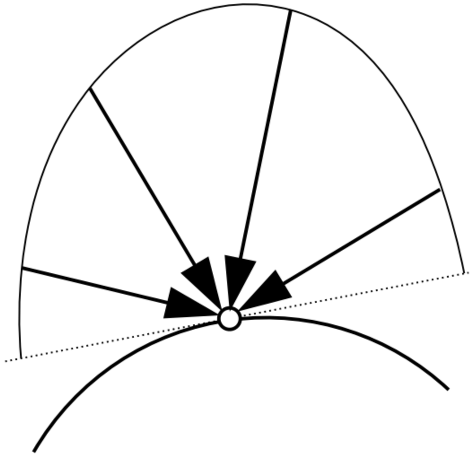
"Radiant Intensity"

$$I(\omega) = \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}$$

power per unit solid angle

(功率角密度)

unit: W/sr or lm/sr



Light Falling On A Surface

"Irradiance"

$$E(p) = \frac{\mathrm{d}\Phi(p)}{\mathrm{d}A}$$

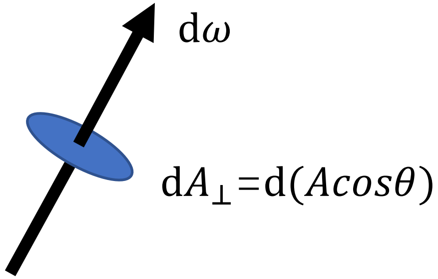
power per unit area

incident on a surface point

（功率面密度）

unit:$\mathrm{W/m^2}$ or $\mathrm{lm/m^2}$

Rendering is all about radiance computation!



Light Traveling Along A Ray

"Radiance"

$$L(p,\omega) = \frac{\mathrm{dI}(\omega)}{\mathrm{d}(A\cos\theta)} = \frac{\mathrm{d}E(p)}{\mathrm{d}\omega\cos\theta} = \frac{\mathrm{d}^2\Phi(p)}{\mathrm{d}\omega\mathrm{d}(A\cos\theta)}$$

the quantity carried by a ray

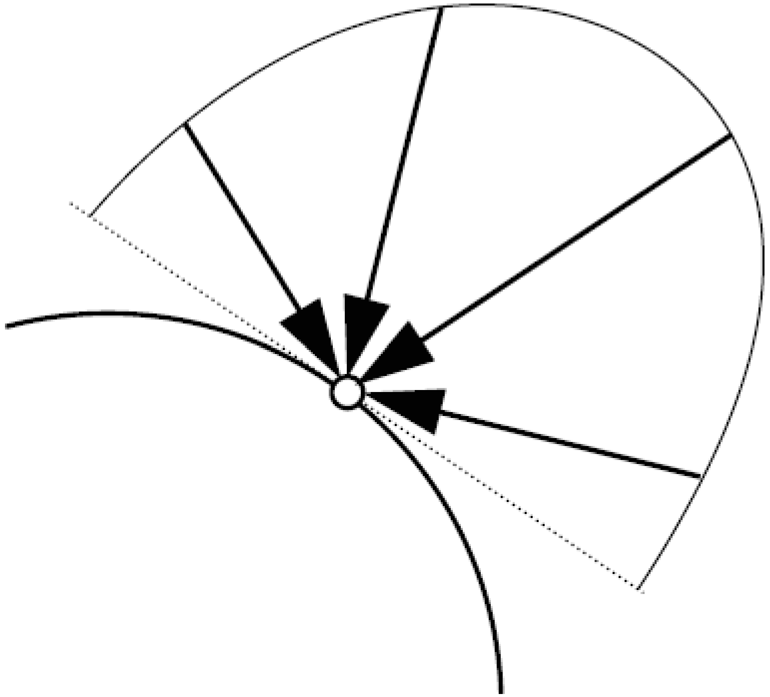
功率角密度的投影面密度

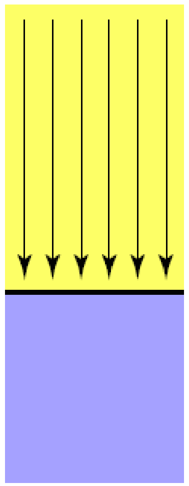
功率投影面密度的角密度

描述某场景的光分布的最基础"场量"

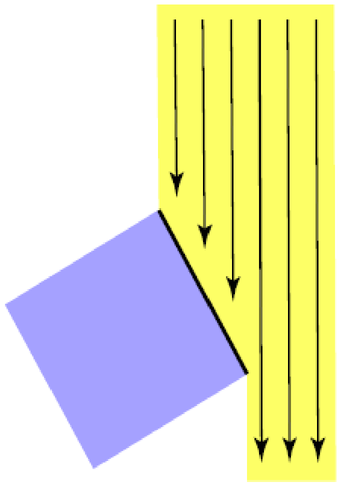
Irradiance

- Irradiance (辐照度)
  - The power per unit area incident on a surface point.
  - $E(x) \equiv \frac{\mathrm{d}\Phi(x)}{\mathrm{d}A}$
  - Lambert's Cosine Law
    - Irradiance at surface is proportional to cosine of angle between light direction and surface normal.

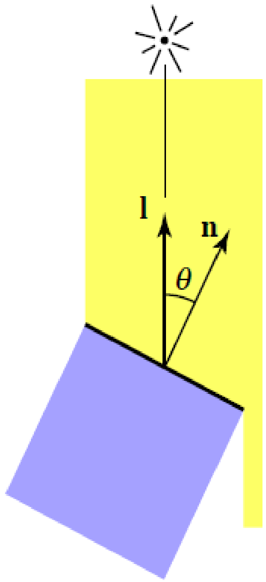




$$E = \frac{\Phi}{A}$$



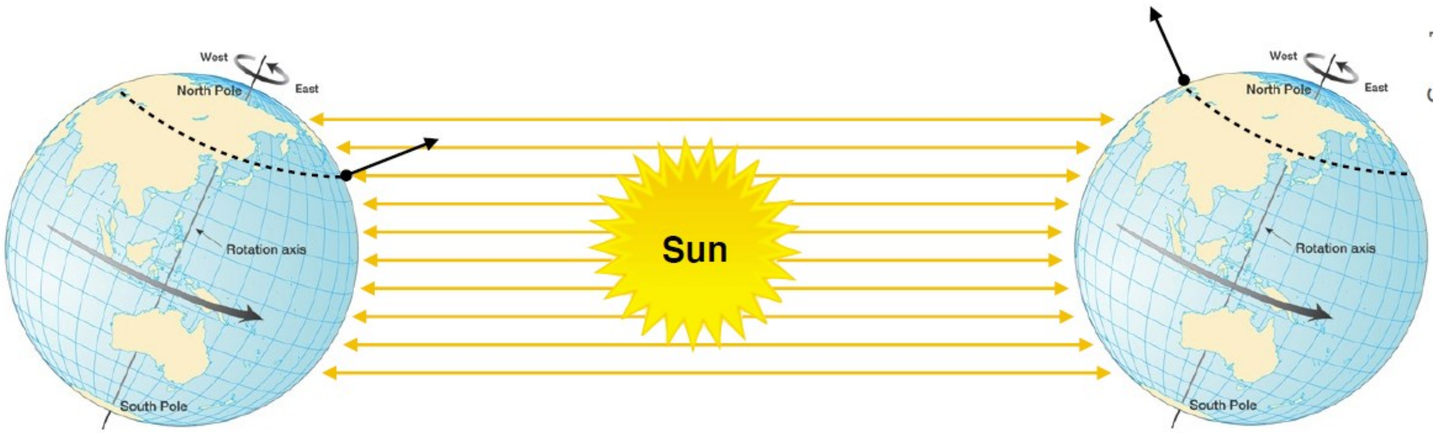
$$E = \frac{\Phi}{2A}$$



$$E = \frac{\Phi}{A}\cos\theta = \frac{\Phi}{A}(\boldsymbol{l} \cdot \boldsymbol{n})$$

Irradiance

- Seasons

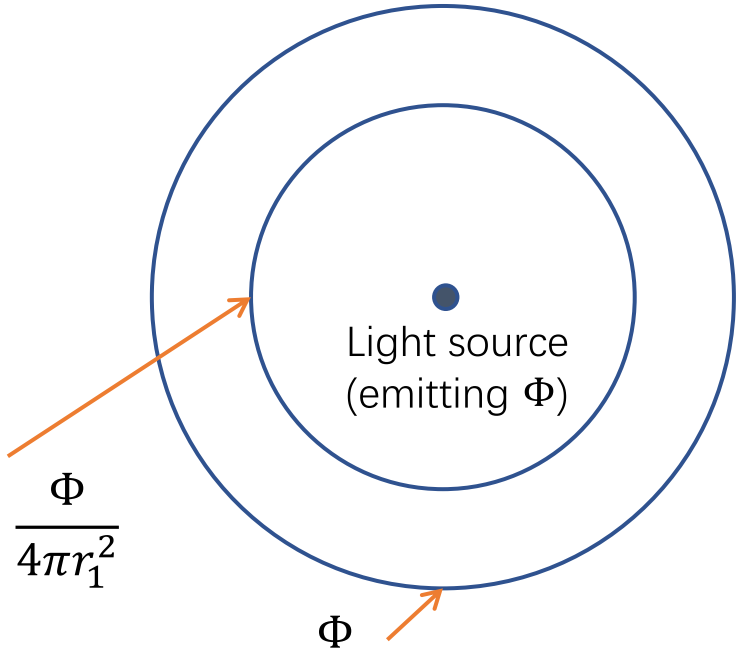


Summer
(smaller θ)

Winter
(greater θ)

Seasons in Northern hemisphere

- Radiant exitance (Irradiance) Falloff
  - Strength of Radiant exitance (Irradiance) decays in proportion to the squared distance

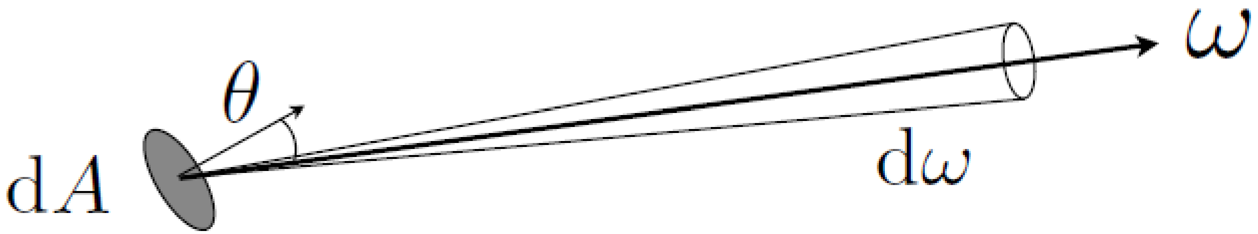


$$E_1 = \frac{\Phi}{4\pi r_1^2}$$

$$E_2 = \frac{\Phi}{4\pi r_2^2}$$

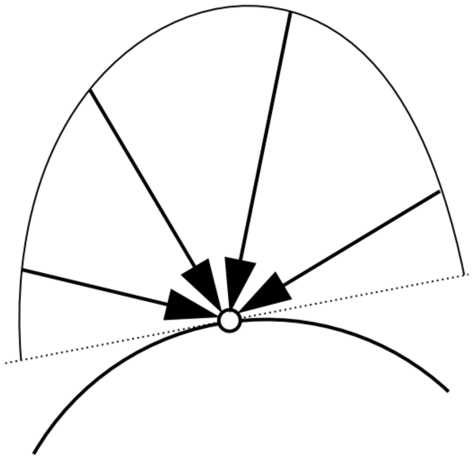
Radiance

- Radiance (辐亮度，辐射度)
  - The radiance (luminance) is the power emitted, reflected, transmitted or received by a surface, per unit solid angle, per projected unit area.
  - $L(p,\omega) \equiv \frac{\mathrm{d}^2\Phi(p,\omega)}{\mathrm{d}\omega \mathrm{d}A \cos\theta}$
    - $\cos\theta$ accounts for the projected surface area
  - Relation with other quantities
    - Irradiance: power per projected unit area → Radiance: Irradiance per solid angle
    - Intensity: power per solid angle → Radiance: Intensity per projected unit area



Incident Radiance

- Irradiance: power per projected unit area → Radiance: Irradiance per solid angle

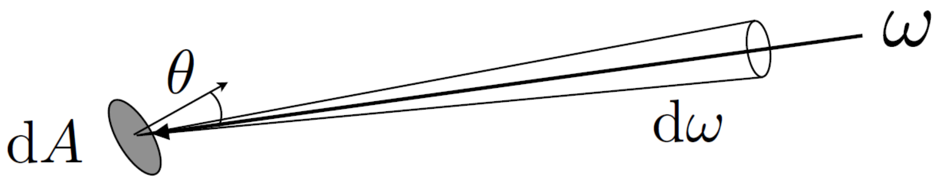


Light Falling On A Surface

"Irradiance"

$$E(p) = \frac{\mathrm{d}\Phi(p)}{\mathrm{d}A_{\perp}}$$

power per unit area
incident on a surface point
（功率面密度）
unit: $\mathrm{W/m^2}$ or $\mathrm{lm/m^2}$



$$L_i(p,\omega) = \frac{\mathrm{d}E(p,\omega)}{\cos\theta\,\mathrm{d}\omega}$$

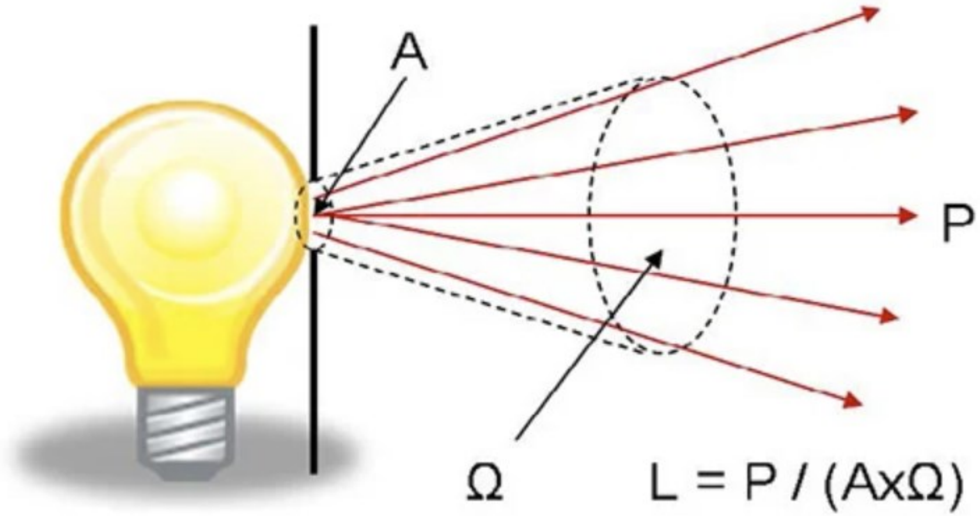
$$E(p) = \int_{H^2} L_i(p,\omega)\cos\theta\,\mathrm{d}\omega$$

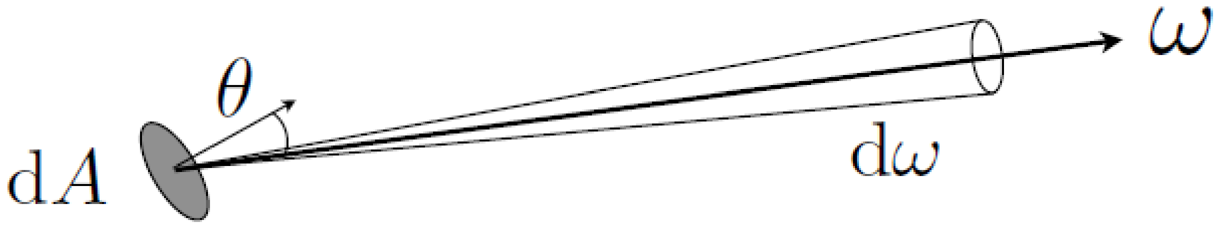
Unit hemisphere: H^2

Irradiance: total power received by area $\mathrm{d}A$
Radiance: power received by area $\mathrm{d}A$ from "direction" $\mathrm{d}\omega$

Exiting Radiance

- Intensity: power per solid angle → Radiance: Intensity per projected unit area





$$L(p,\omega) \equiv \frac{\mathrm{d}I(p,\omega)}{\mathrm{d}A\cos\theta}$$

$$I(\omega) = \int_{\mathrm{A}} L_i(p,\omega)\cos\theta\,\mathrm{d}A$$

Radiance

- Incident radiance (入射辐亮度)
  - Incident radiance is the irradiance per unit solid angle arriving at the surface.
    - i.e., it is the light arriving at the surface along a given ray (point on surface and incident direction).
  - $L(p,\omega) = \dfrac{\mathrm{d}E(p)}{\mathrm{d}\omega \cos\theta}$; $E(p) = \int_{H^2} L_i(p,\omega) \cos\theta \, \mathrm{d}\omega$

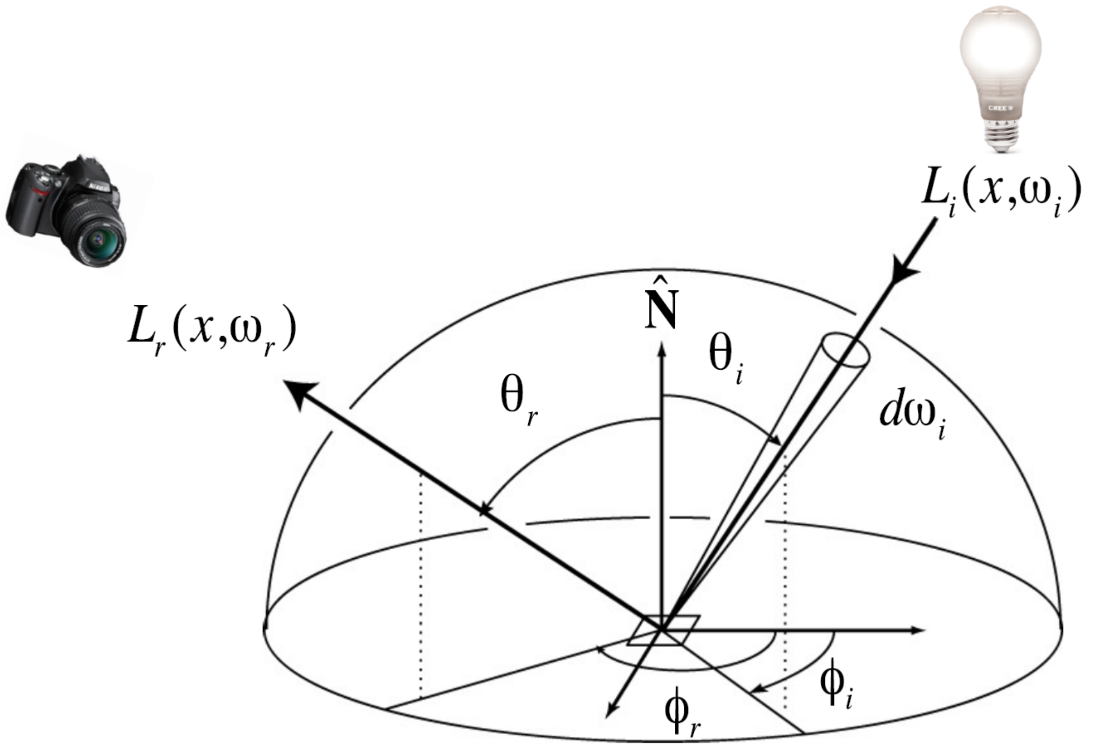
- Exiting radiance (出射辐亮度)
  - Exiting surface radiance is the intensity per unit projected area leaving the surface.
    - e.g., for an area light it is the light emitted along a given ray (point on surface and exit direction).
  - $L(p,\omega) \equiv \dfrac{\mathrm{d}I(p,\omega)}{\mathrm{d}A \cos\theta}$; $I(\omega) = \int_{\mathrm{A}} L_i(p,\omega) \cos\theta \, \mathrm{d}A$

Last Question: How?

- Path Tracing:
  - Radiometry (辐射度量学), a measurement system for illumination
  - **Integral** Light Transport **Equations**
    - The reflection equation
    - The rendering equation
  - **Probability** and Monte Carlo Integration
  - Algorithm

Integral Light Transport **Equations**

- The Reflection Equation

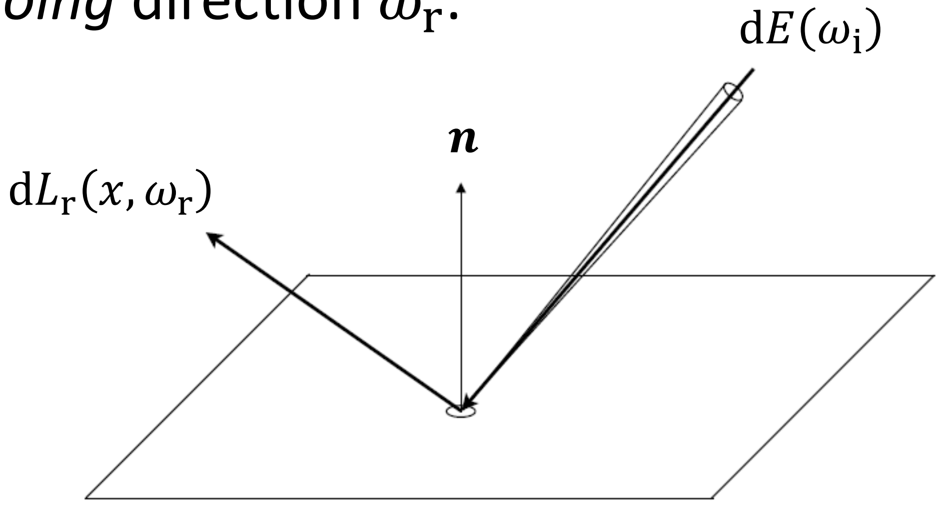


Unit hemisphere: H^2

$$L_r(\mathrm{p}, \omega_r) = \int_{H^2} f_r(\mathrm{p}, \omega_i \to \omega_r)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

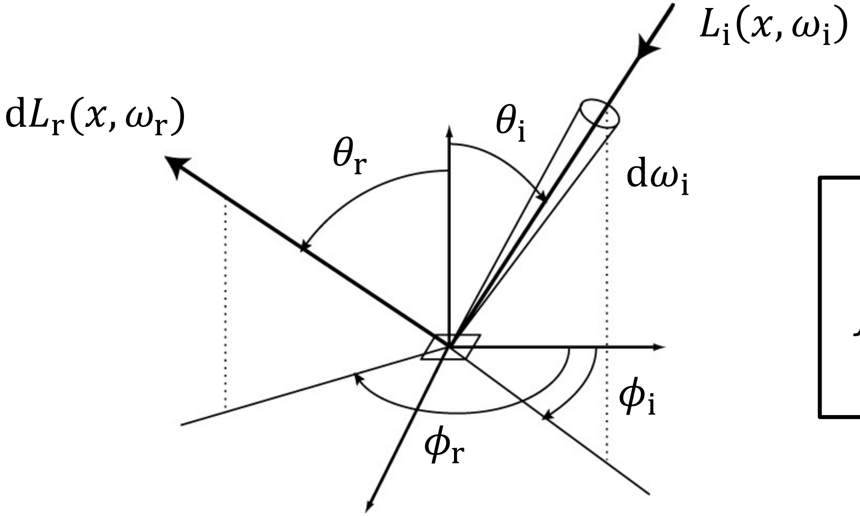
Bidirectional Reflectance Distribution Function

- Reflection at a Point
  - At $\mathrm{d}A$, Radiance from *incoming* direction ω_i turns into the irradiance $\mathrm{d}E$
  - $\mathrm{d}E$ then become the radiance $\mathrm{d}L_\mathrm{r}$ to some *outgoing* direction ω_r.



- The Bidirectional Reflectance Distribution Function (BRDF, 双向反射分布函数)

  How much light is reflected into ω_r from ω_i

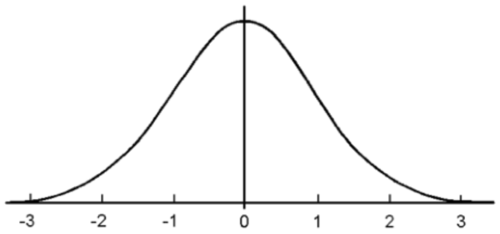


$$f_\mathrm{r}(\omega_\mathrm{i} \to \omega_\mathrm{r}) = \frac{\mathrm{d}L_\mathrm{r}(\omega_\mathrm{r})}{\mathrm{d}E_\mathrm{i}(\omega_\mathrm{i})} = \frac{\mathrm{d}L_\mathrm{r}(\omega_\mathrm{r})}{L_\mathrm{i}(\omega_\mathrm{i}) \cos\theta_\mathrm{i} \, \mathrm{d}\omega_\mathrm{i}}$$

Review: Probabilities

- Continuous variable X and Probability density function $p(x)$
  - $X \sim p(x)$

  - Understanding: randomly pick an X → more likely to be a number closer to 0 (in this case)

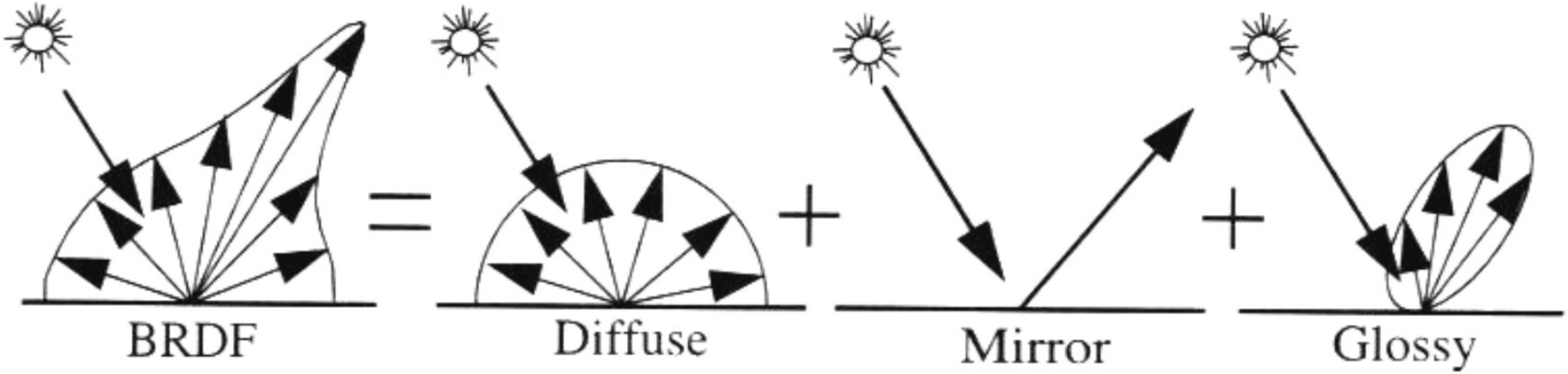


- Conditions on $p(x)$
  - $p(x) \geq 0$
  - $\int p(x)\mathrm{d}x = 1$
- Expected value of X
  - $E(x) = \int xp(x)\mathrm{d}x$

Bidirectional Reflectance Distribution Function

Description of visual surface appearance:

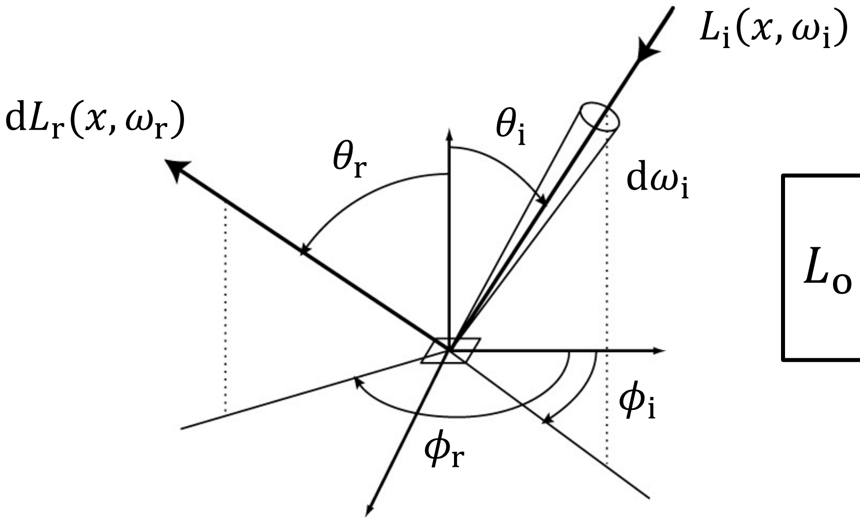
- Ideal specular reflection (Reflection law, Mirror)
- Glossy reflection (Directional diffuse, Shiny surfaces)
- Ideal diffuse reflection (Lambert's law, Matte surfaces)



The Rendering Equation

- The reflection equation
  - $L_r(p, \omega_r) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_r) L_i(p, \omega_i) \cos\theta_i \, d\omega_i$
    - Reflected radiance depends on incoming radiance,
    - But incoming radiance depends on reflected radiance (at another point in the scene)

- The rendering equation (rewritten reflection equation)



L_i at p coincides with L_o at some other point (just as the plain ray-tracing method)

emission term

$\cos\theta_i$

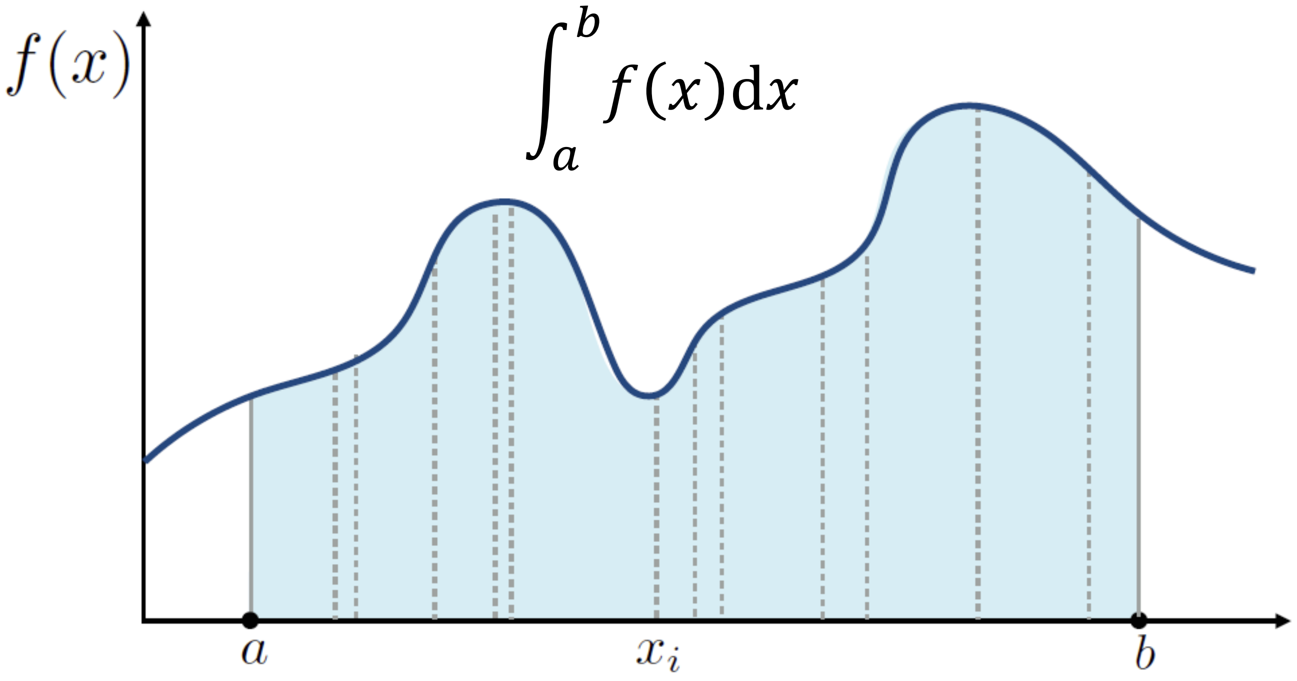
$$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{H^2} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o)(n \cdot \omega_i) d\omega_i$$

Last Question: How?

- Path Tracing:
  - Radiometry (辐射度量学), a measurement system for illumination
  - Integral Light Transport Equations
    - The reflection equation
    - The rendering equation
  - **Probability and Monte Carlo Integration**
  - Algorithm

Monte Carlo Integration

- How to numerically estimate the integral of a function?
  - Averaging random samples of the function's value.



- Definite integral

$$\int_a^b f(x)\mathrm{d}x$$

- Random variable

$$X_i \sim p(x)$$

- Monte Carlo estimator

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

Monte Carlo Integration

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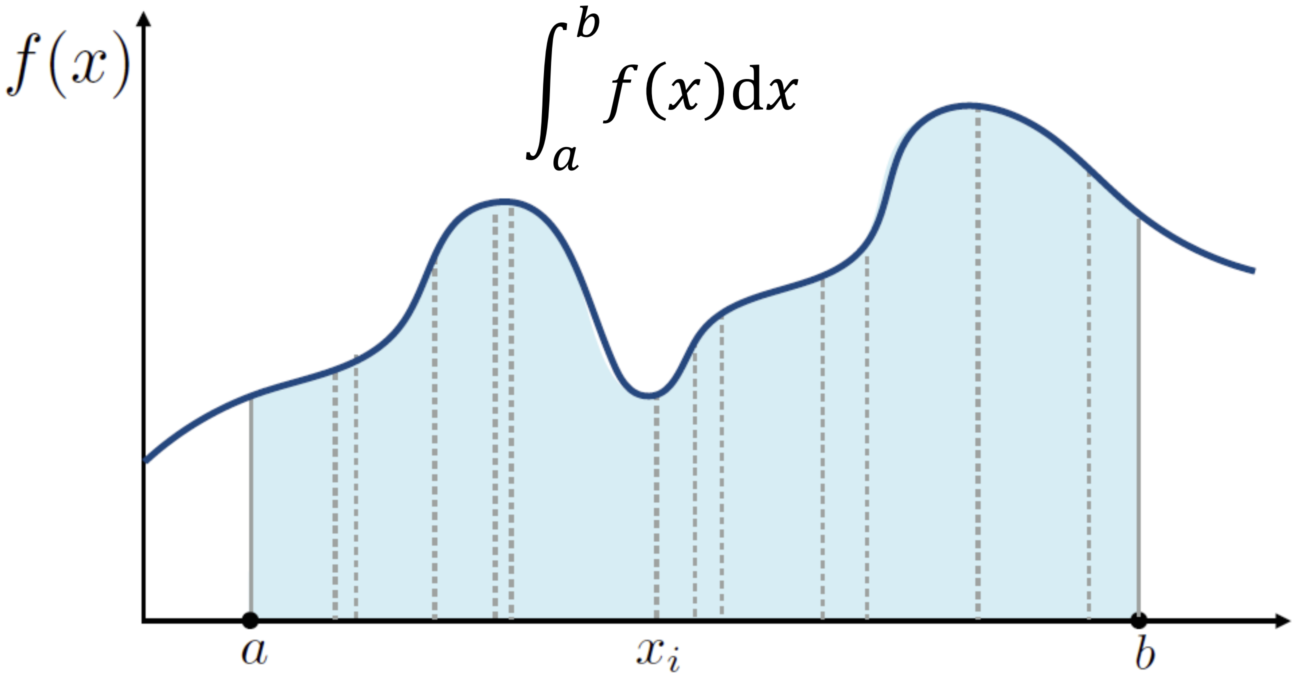
  - Averaging random samples of the function's value.



uniform sampling

- Definite integral

$$\int_a^b f(x)\mathrm{d}x$$

- Random variable

$$X_i \sim p(x) \left(= \frac{1}{b-a} \right)$$

- Monte Carlo estimator

$$F_N = \frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \left(= \frac{b-a}{N}\sum_{i}^{N} f(X_i) \right)$$

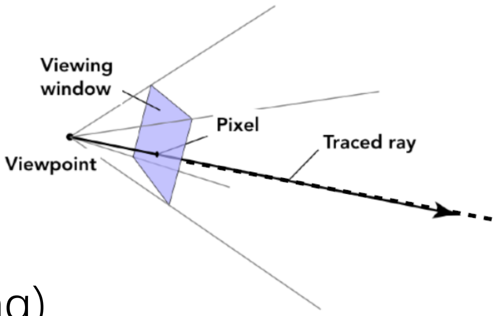
A Simple Monte Carlo Solver

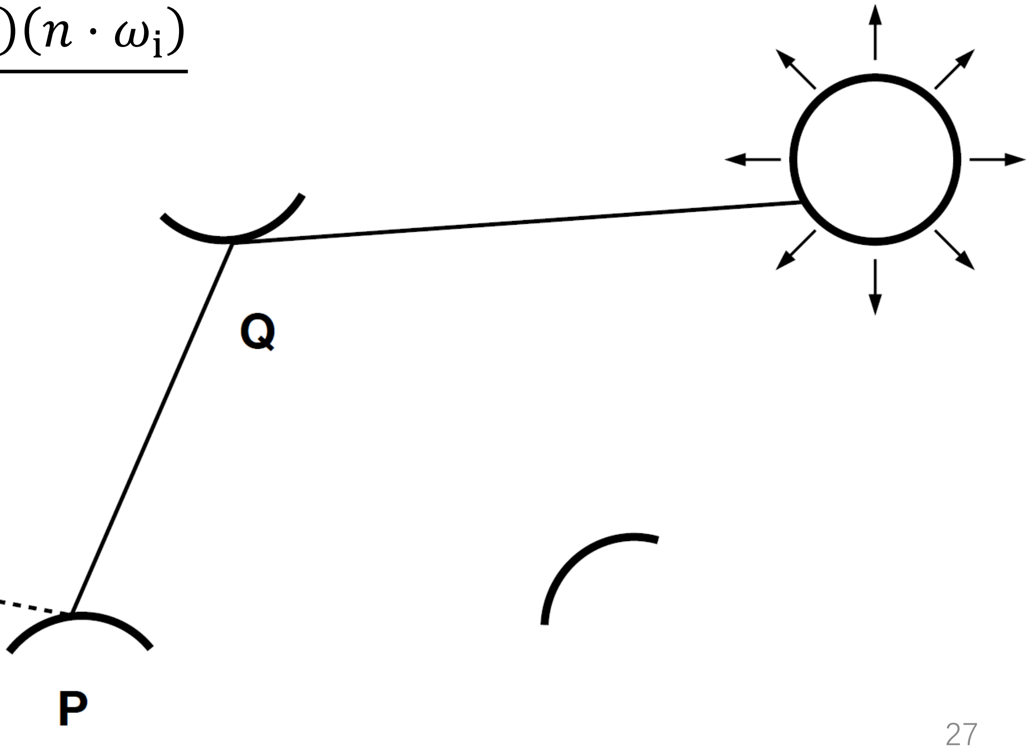
- A simple Monte Carlo solver to the rendering equation

$$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{\Omega^+} L_i(p, \omega_i) f_r(p, \omega_i, \omega_o)(n \cdot \omega_i) \mathrm{d}\omega_i$$

➔

$$L_o(p, \omega_o) = L_e(p, \omega_o) + \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \omega_i) f_r(p, \omega_i, \omega_o)(n \cdot \omega_i)}{p(\omega_i)}$$





(Just like plain ray tracing)

Last Question: How?

- Path Tracing:
  - Radiometry (辐射度量学), a measurement system for illumination
  - Integral Light Transport Equations
    - The reflection equation
    - The rendering equation
  - Probability and Monte Carlo Integration
  - **Algorithm**

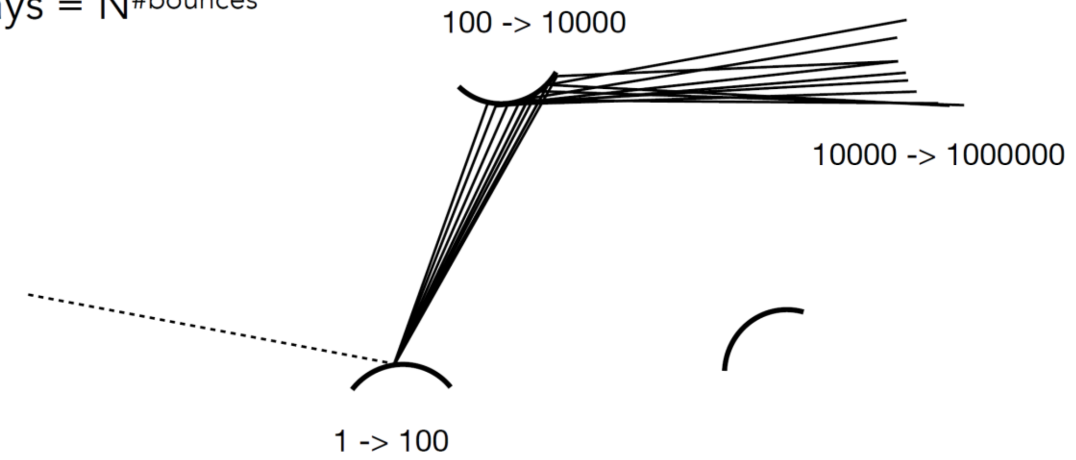
A Simple Monte Carlo Solver

- A simple Monte Carlo solver to the rendering equation

```
shade(p, wo)
    Randomly choose N directions wi~pdf
    Lo = 0.0
    For each wi
        Trace a ray r(p, wi)
        If ray r hit the light
            Lo += (1 / N) * L_i * f_r * cosine / pdf(wi)
        Else If ray r hit an object at q
            Lo += (1 / N) * shade(q, -wi) * f_r * cosine / pdf(wi)
    Return Lo
```

Multiple recursive (多重递归)

#rays = $N^{\#bounces}$



- Problem: exponential explosion

Path Tracing

- Path-tracing solver to the rendering equation

Main difference from the simple alg.:
Only trace **ONE** light

```
shade(p, wo)
    Manually specify a probability P_RR
    Randomly select ksi in a uniform dist. in [0, 1]
    If (ksi > P_RR) return 0.0;

    Randomly choose ONE direction wi~pdf(w)
    Trace a ray r(p, wi)
    If ray r hit the light
        Return L_i * f_r * cosine / pdf(wi) / P_RR
    Else If ray r hit an object at q
    Return shade(q, -wi) * f_r * cosine / pdf(wi) / P_RR
```

Russian roulette (俄罗斯轮盘赌)

- To stop algorithm
- The light does not stop bouncing indeed
  - Directly limiting bounces is wrong

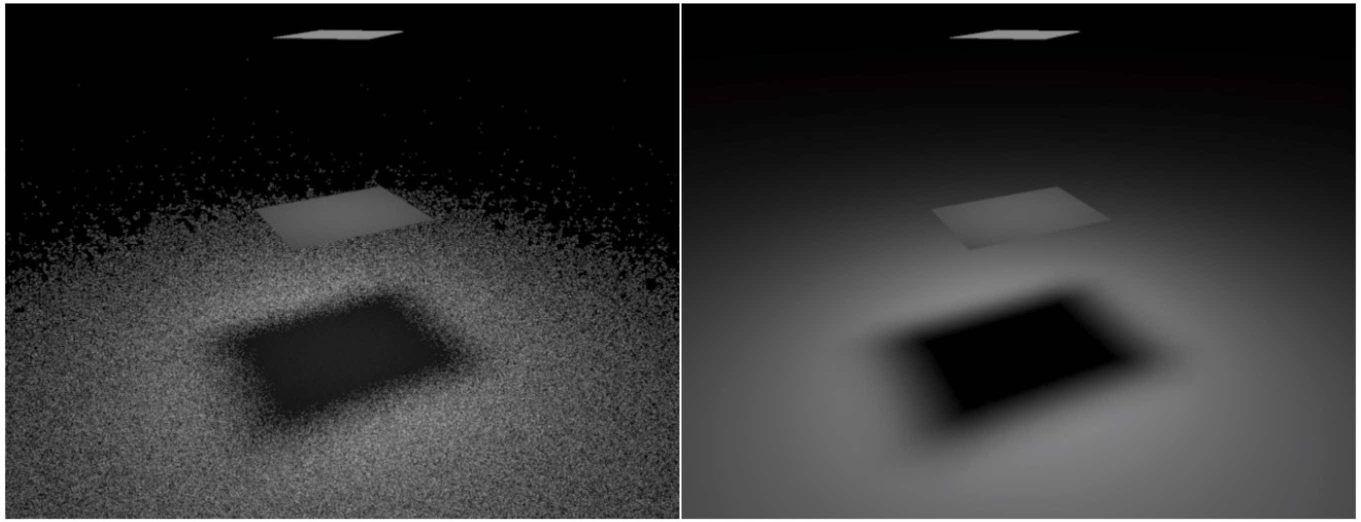
Tail-recursive (尾递归)

Path Tracing

- Ray generation
  - If we sample ONE light per pixel, the result is noisy (the left image)
    - Solution: sample many lights per pixel, as anti-aliasing in ray tracing

```
ray_generation(camPos, pixel)
    Uniformly choose N sample positions within the pixel
    pixel_radiance = 0.0
    For each sample in the pixel
        Shoot a ray r(camPos, cam_to_sample)
        If ray r hit the scene at p
            pixel_radiance += 1 / N * shade(p, sample_to_cam)
    Return pixel_radiance
```

Low SPP
(samples per pixel)



High SPP

Cutting the Number of Bounces

- Direct illumination
  - (Zero bounce)

Cutting the Number of Bounces

- Global illumination
  - (One bounce)

Cutting the Number of Bounces

- Global illumination
  - (Two bounces)

Cutting the Number of Bounces

- Global illumination
  - (Four bounces)

Cutting the Number of Bounces

- Global illumination
  - (Eight bounces)

Cutting the Number of Bounces

- Global illumination
  - (16 bounces)
- Conclusion
  - Higher brightness
  - ➔ (often means)
  - Higher accuracy

Is Path Tracing Correct?

- Yes, almost 100% correct, a.k.a. PHOTO-REALISTIC

Photo

Path-traced global illumination

The Cornell box — http://www.graphics.cornell.edu/online/box/compare.html

Advanced Topics

BRDF,

Other Materials,

Importance Sampling,

Other Rendering Methods,

Bidirectional Reflectance Distribution Function

- Measurement by gonioreflectometer (角反射仪)
  - An image-based approach
  - General pipeline

```
for each outgoing direction ωo
    move light to illuminate surface with a thin beam from ωo
    for each incoming direction ωi
        move sensor to be at direction ωi from surface
        measure incident radiance
```

- Efficiency improvement
  - Isotropic surfaces reduce dimensionality from 4D to 3D
  - Reciprocity reduces the number of measurements by half
  - Clever optical systems…?

Isotropic: $f_r(\theta_i, \phi_i; \theta_r, \phi_r) = f_r(\theta_i, \theta_r, \phi_r - \phi_i)$

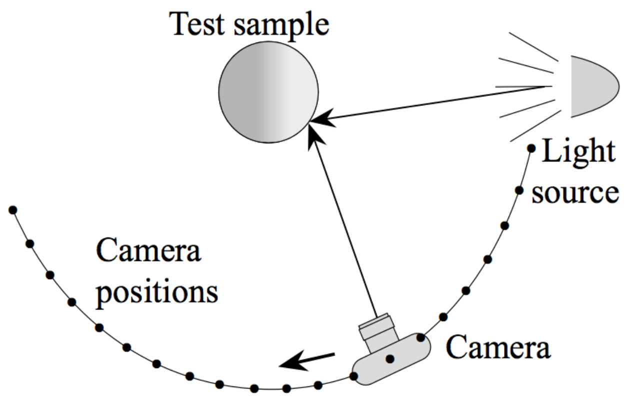
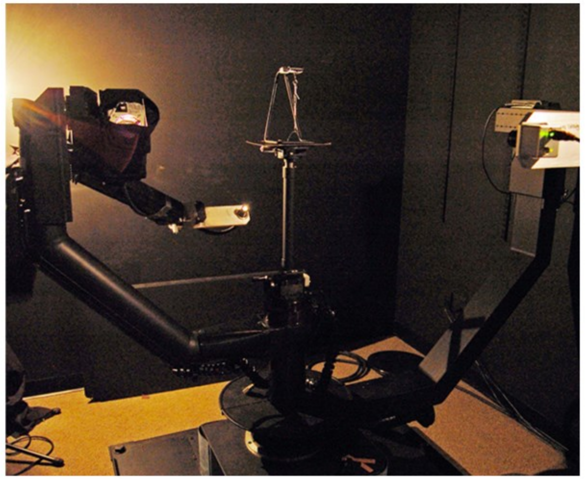


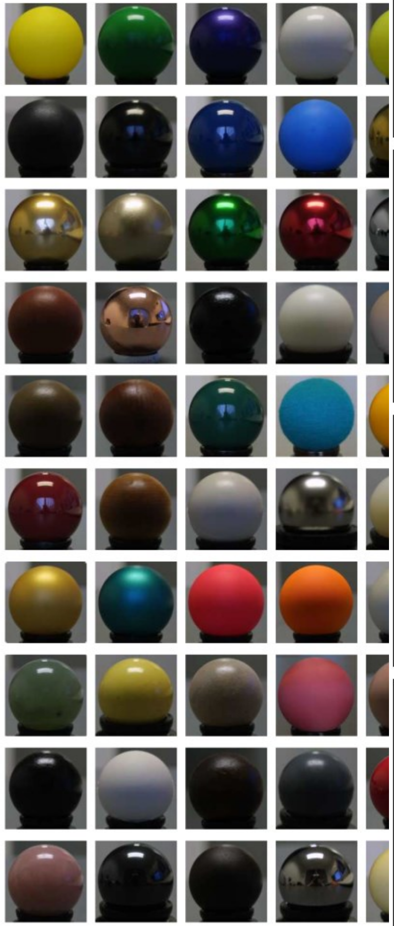
Image-Based BRDF Measurement Including Human Skin [Marschner et al. 1999]

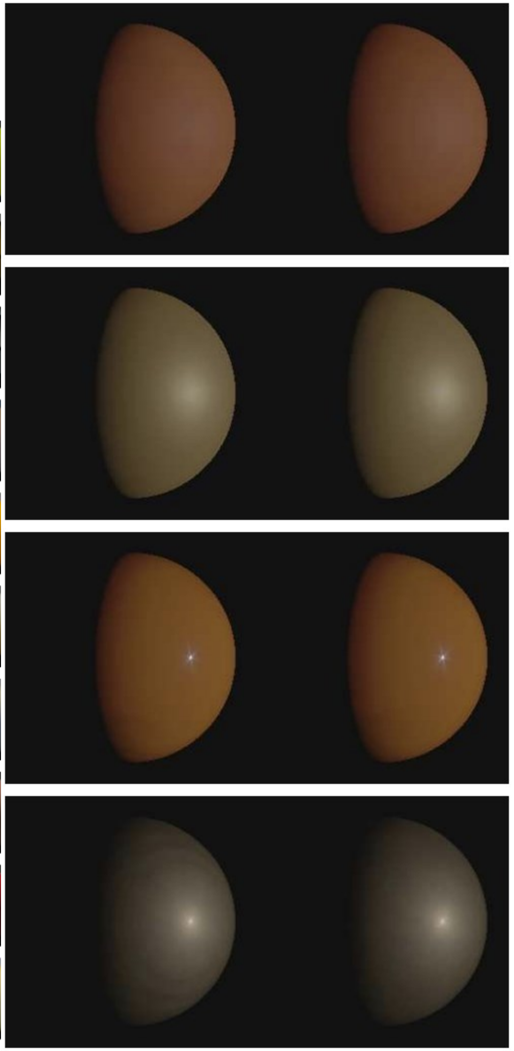


Spherical gantry at UCSD

Bidirectional Reflectance Distribution Function

- Tabular representation
  - Store regularly-spaced samples in $(\theta_\mathrm{i}, \theta_\mathrm{o}, |\phi_\mathrm{i} - \phi_\mathrm{o}|)$
    - Reparameterize angles to better match specularities
    - Generally need to resample measured values to table
    - Very high storage requirements
  - A data-driven reflectance model [Matusik et al. 2003]
    - θ_i: 90 samples
    - θ_o: 90 samples
    - $|\phi_\mathrm{i} - \phi_\mathrm{o}|$: 180 samples
    - Thus 1,458,000 samples per material
      - (per sphere in the right)

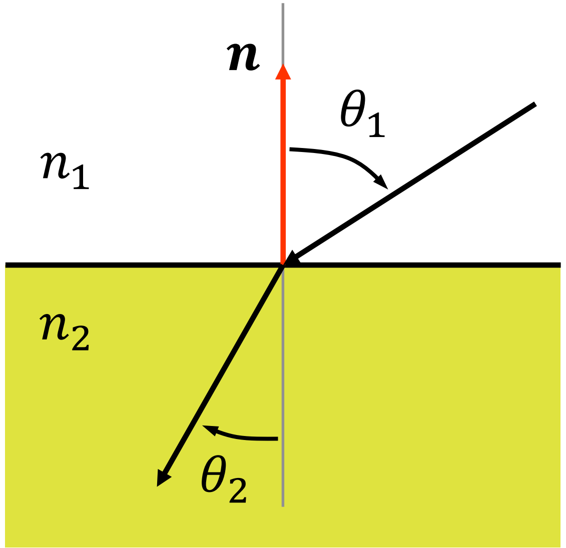




Bidirectional Reflectance Distribution Function

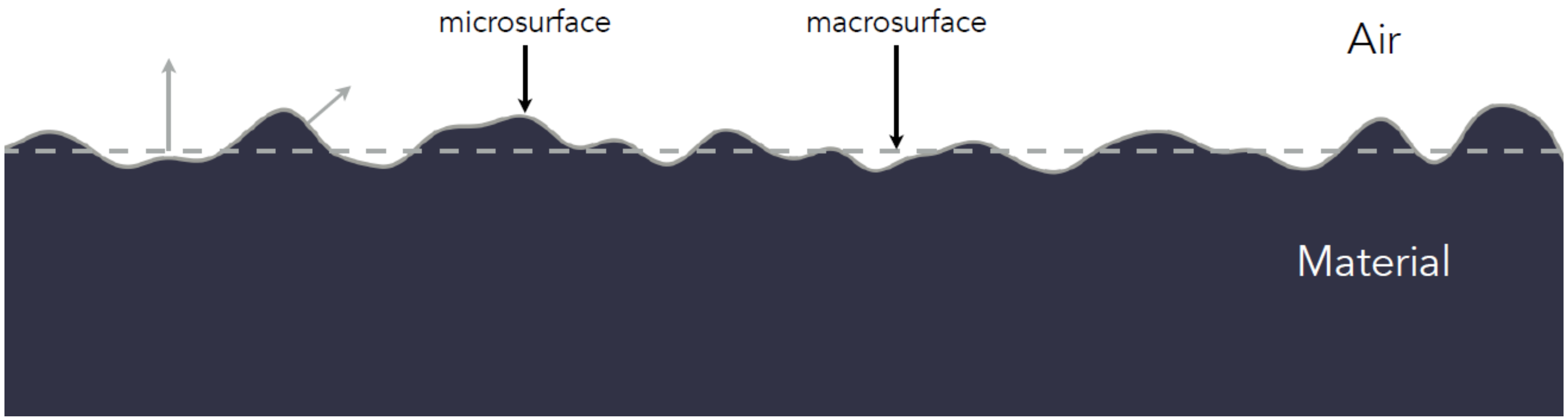
- Snell's Law and Fresnel Term
  - $n_1 \sin\theta_1 = n_2 \sin\theta_2$
  - Reflectance (反射率) R depends on incident angle
    - The brightness of shadows varies as grazing angle increases (the right figures)
    - Non-linear approximation of reflectance functions [Lafortune et al. 1997]
  - Accurate Fresnel term: need to consider polarization
    - $R_\mathrm{s} = \left|\frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}\right|^2$
    - $R_\mathrm{p} = \left|\frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right|^2$
    - $R_\mathrm{eff} = \frac{1}{2}(R_\mathrm{s} + R_\mathrm{p})$
  - Approximate: Schlick's approximation
    - $R_0 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$
    - $R(\theta_1) = R_0 + (1 - R_0)(1 - \cos\theta_1)^5$

$$\theta_1 \to 0, R \to \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2; \quad \theta_1 \to \frac{\pi}{2}, R \to 1;$$

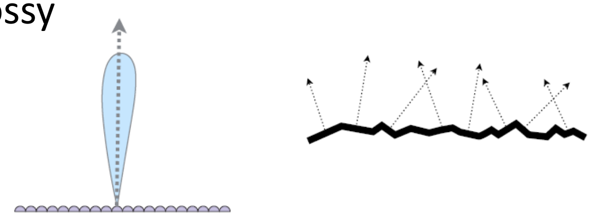


Bidirectional Reflectance Distribution Function

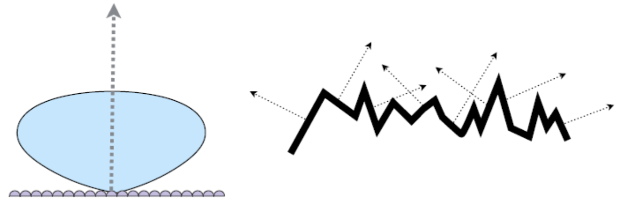
- Microfacet Model (微表面模型)
  - Rough surface
    - Macroscale: flat & rough
    - Microscale: bumpy & specular
  - Individual elements of surface act like mirrors
    - Known as Microfacets
    - Each microfacet has its own normal



- Microfacet BRDF
  - Key: the distribution of microfacets’ normal
    - Concentrated ➔ glossy



    - Spread ➔ diffuse

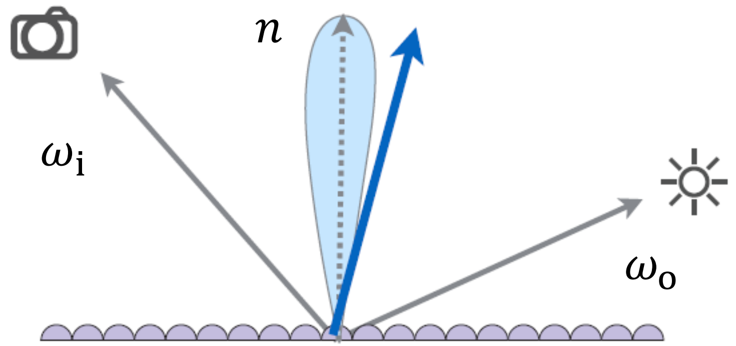


Bidirectional Reflectance Distribution Function

- Microfacet BRDF
  - Key: the distribution of microfacets' normal

Fresnel term shadowing-masking term distribution of normals

$$f(\omega_\mathrm{i}, \omega_\mathrm{o}) = \frac{F(\omega_\mathrm{i}, h)\,G(\omega_\mathrm{i}, \omega_\mathrm{o}, h)\,D(h)}{4(n \cdot \omega_\mathrm{i})(n \cdot \omega_\mathrm{o})}$$



[Autodesk Fusion 360]
How to acquire these images?
(explained in the follows)

Advanced Appearance Modeling

- Non-surface models
  - Participating media
  - Hair / fur / fiber (BCSDF)
  - Granular material
- Surface models
  - Translucent material (BSSRDF)
  - Cloth
  - Detailed material (non-statistical BRDF)

[Yan et al. 2015]

[Yan et al. 2014, 2016]

[Meng et al. 2015]

Accelerating Path Tracer

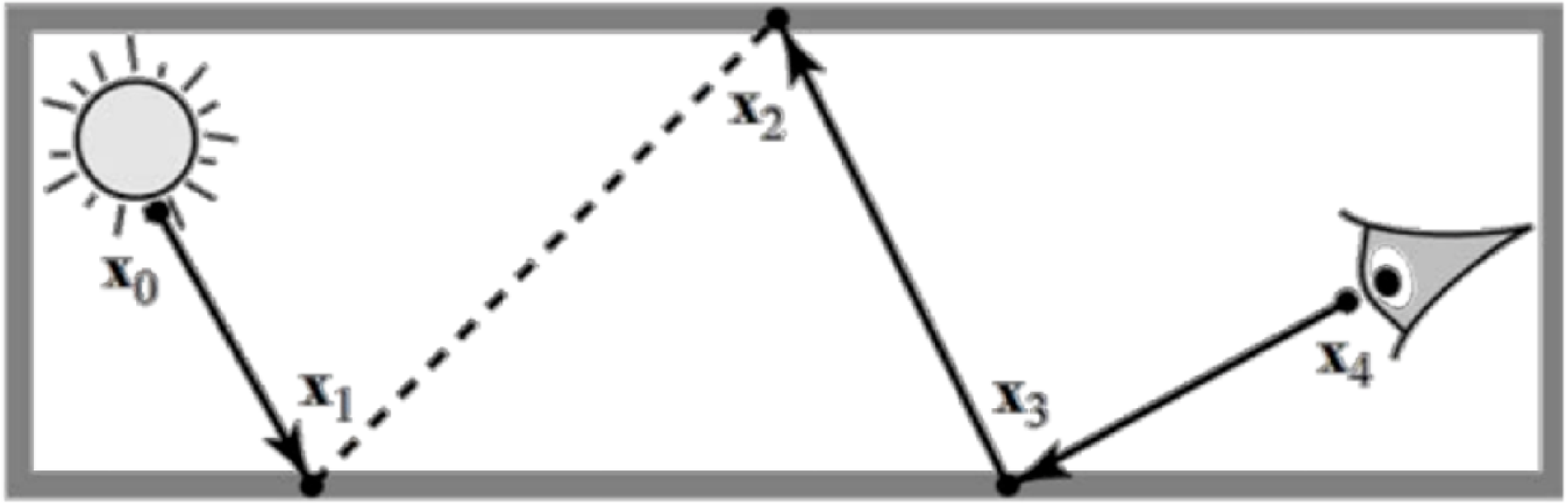
- $L_\mathrm{o}(p, \omega_\mathrm{o}) = L_\mathrm{e}(p, \omega_\mathrm{o}) + \frac{1}{N}\sum_{i=1}^{N} \frac{L_\mathrm{i}(p,\omega_\mathrm{i}) f_\mathrm{r}(p,\omega_\mathrm{i},\omega_\mathrm{o})(n\cdot\omega_\mathrm{i})}{p(\omega_i)}$
  - How to choose $p(\omega_i)$?
  - Is there any sampling more efficient than uniform one?
- Importance sampling (重要性采样)
  - Recall: $F_N = \frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$
  - $E[F_N] = \int_a^b \frac{f(x)}{p(x)} p(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$
  - What if $\frac{f(x)}{p(x)} \equiv \lambda$?
    - $F_N = \lambda$
    - It suggests us to
      - sample more points where $f(x)$ is large.
- Idea: the following expectations are equivalent
  - $E[f(X)] = \int f(x)p(x)\mathrm{d}x$
    - $X \sim p(x)$
  - $E\left[\frac{p(X)}{q(X)} f(X)\right] = \int f(x) \frac{p(x)}{q(x)} q(x)\mathrm{d}x$
    - $X \sim q(x)$

Advanced Light Transport

- Unbiased light transport methods
  - Bidirectional path tracing (BDPT)
  - Metropolis light transport (MLT)
- Biased light transport methods
  - Photon mapping
  - Vertex connection and merging (VCM)
  - Instant radiosity (VPL / many light methods)

Bidirectional path tracing (BDPT), unbiased

- Traces sub-paths from both the camera and the light
- Connects the endpoints from both sub-paths



[Veach 1997]

Bidirectional path tracing (BDPT), unbiased

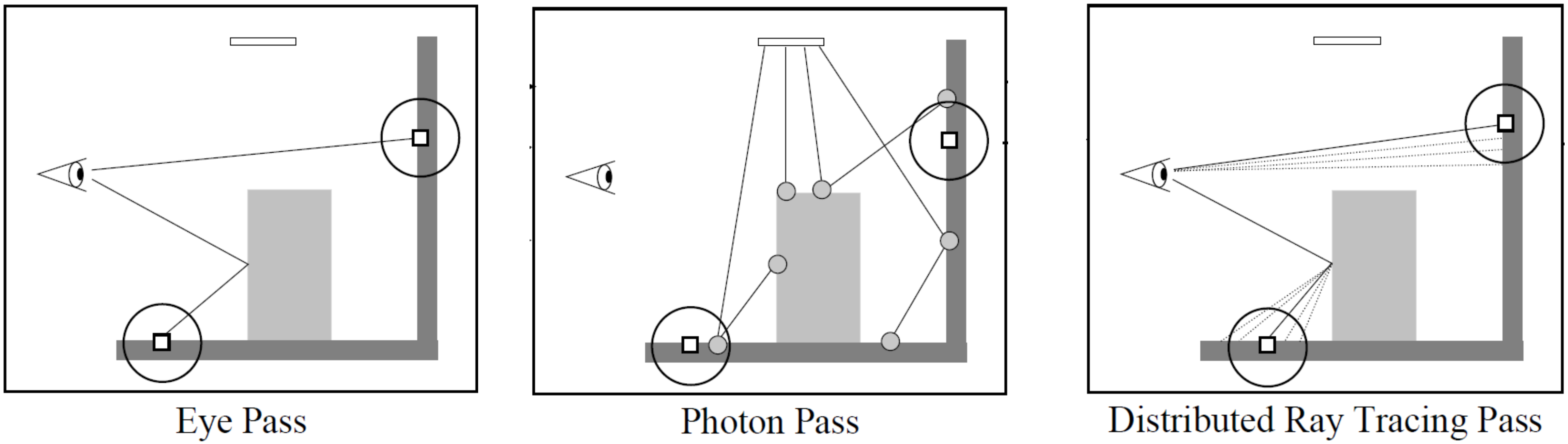
- Traces sub-paths from both the camera and the light
- Connects the endpoints from both sub-paths

(a) Path tracer, 32 samples/pixel

(b) Bidirectional path tracer, 32 samples/pixel

Photon Mapping, biased

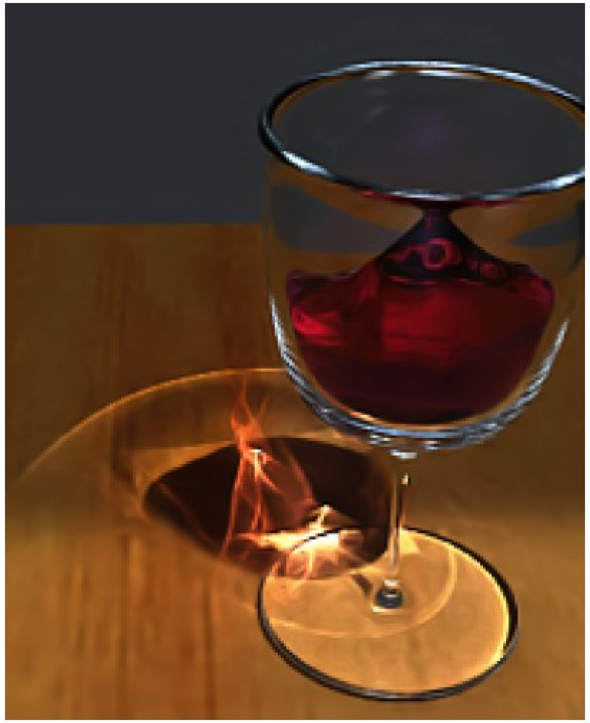
- Stage 1 — photon tracing
- Stage 2 — photon collection (final gathering)



Photon Mapping, biased

- Very good at handling Specular-Diffuse-Specular (SDS) paths and generating **caustics**

(a) 聚光灯



(b) 平行光

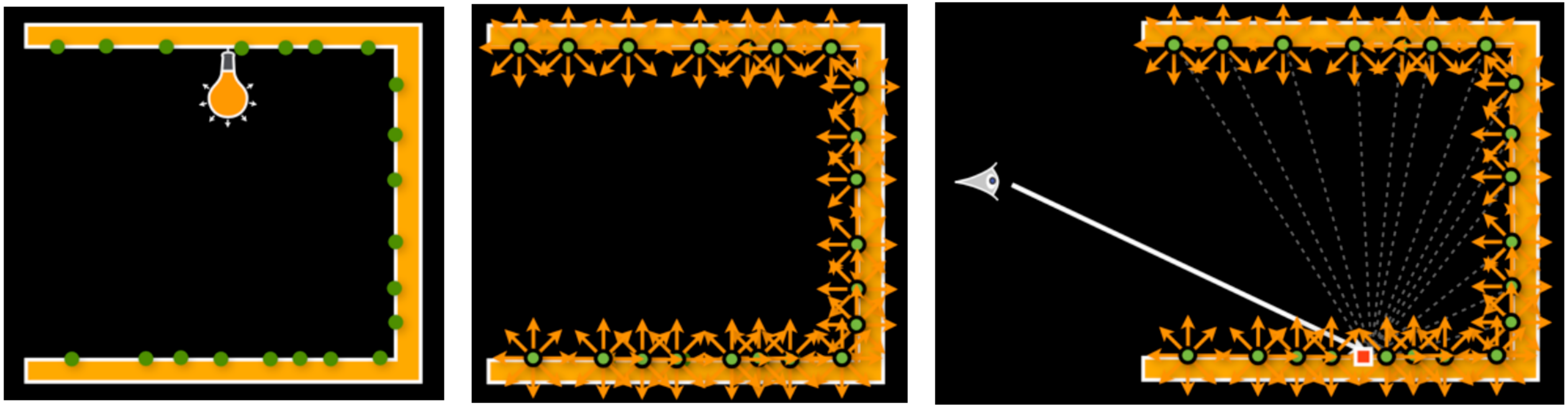
(c) 面光源

Instant Radiosity

Sometimes also called many-light approaches

Key idea: Lit surfaces can be treated as light sources

- Pros: fast and usually gives good results on diffuse scenes
- Cons: difficult to handle reflection/refraction/glossy…



[image courtesy of Derek N.]

Instant Radiosity

- B_i (Radiosity/ Radiant exitance, unit W/m^2)

$$B_i = E_i + \rho_i \sum_{s_j} B_i F_{ji}$$

  - F_{ji}, Form Factor, $\frac{\text{flux from face j to face i}}{\text{flux of face j}}$

$$F_{ji} = \frac{1}{A_i} \int_{x \in s_i} \int_{x' \in s_j} \frac{1}{\pi} \mathrm{G}(x', x) \, \mathrm{d}A(x') \, \mathrm{d}A(x)$$

$$G(x', x) = \frac{\cos\theta \cos\theta'}{\|x' - x\|^2} \mathrm{v}(x', x) \qquad \mathrm{v}(x', x) = \begin{cases} 1, \text{ i j are visible to each other} \\ 0, \text{ otherwise} \end{cases}$$

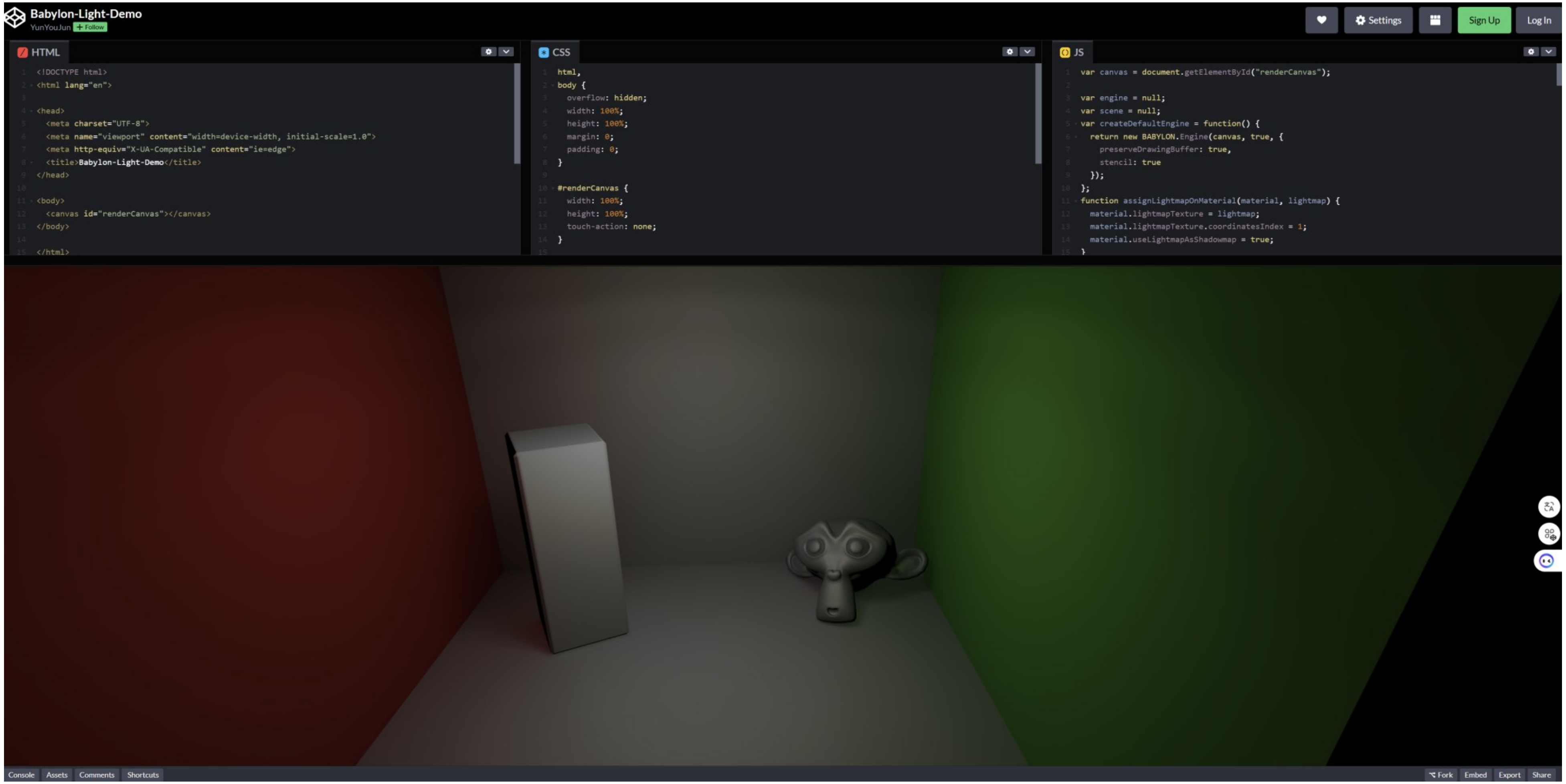
  - ρ_i, Reflectivity of face i

    a parameter

$$\begin{bmatrix} 1 - \rho_1 F_{11} & -\rho_1 F_{12} & \cdots & -\rho_1 F_{1N} \\ -\rho_2 F_{21} & 1 - \rho_2 F_{22} & \cdots & -\rho_2 F_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_N F_{N1} & -\rho_N F_{N2} & \cdots & 1 - \rho_N F_{NN} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{bmatrix}$$

Instant Radiosity

[Babylon-Light-Demo](https://codepen.io/YunYouJun/pen/VwYMKMy)
https://codepen.io/YunYouJun/pen/VwYMKMy



Conclusion

- Path Tracing
  - Why? Ray tracing Fails to support glossy/diffuse reflection/soft shadows…
  - What? A ray-tracing method following rendering equations
  - How? Monte Carlo Path Tracing

- Advanced Topics
  - *Advanced Appearances*
  - *Importance Sampling*
  - *Other Rendering Methods*